Name:______________________________

Math 8663 Section 01 | *Practice Final Exam* | *November 7, 2019*

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work, answer or there is no justification to a solution, you will receive little or no credit!

1. Let $C[0,1]$ be equipped with the supremum norm. Define the following operator on $C[0,1]$:

$$A = \frac{d}{dx}$$

with domain $\mathcal{D}(A) = C^1[0,1]$ as a subspace of $C[0,1]$. Prove that A has closed graph. Assuming we know A is unbounded on this space, why doesn't this contradict the Closed Graph Theorem?

2. Suppose X and Y are Banach spaces. If $T \in L(X,Y)$ and Ran A is of second category, prove that Ran A is closed.

3. Let X be a Banach space and $A \in L(X,X)$. Suppose A has the property that $A^n = 0$ for some n. Prove that $\sigma(A) = \{0\}$.

4. Let X be a compact space and $f \in C(X)$. Prove that $\sigma(f) = f(X)$.

5. Let $T \in L(X,X)$ where X is a normed space. Prove that $\sigma(T^*) = \overline{\sigma(T)}$. Here the bar stands for complex conjugation.

6. Let $\mathcal{H}$ be a Hilbert space and let A be a self-adjoint(not necessarily bounded) operator defined on $\mathcal{H}$ with densely defined domain in $\mathcal{H}$. Prove that

$$\lim_{t\to\infty} e^{-tA^2} = 0$$

7. Let $T \in L(H,H)$ for some Hilbert space H be a self-adjoint operator and $f \in C(\sigma(T))$. If $f(T) \geq 0$, prove that $f(t) \geq 0$ for all $t \in \sigma(T)$.

8. Let T be either the right or left shift operator on $\ell^2(\mathbb{N})$ and f a holomorphic function on a disk with some radius $r > 1$. Prove that $\sigma(f(T)) = f(\mathbb{D})$. Moreover if T is specifically the right shift operator, prove that $\sigma_p(f(T)) = \emptyset$.